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Safety-Critical Control for Ensemble Systems

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Abstract: In this paper, we derive set constraints for a reduced order model and augment them into a model predictive control (MPC) scheme to ensure safe operation of the large-scale ensemble system. For the control feedback, only the aggregated information of the whole system is required. For the constraint satisfaction, we consider an adaptive tube formulation to characterize the deviation between the reduced order model and the ensemble system. Employing the robust control invariant set, we ensure recursive feasibility and initial feasibility under an easily verifiable condition.

Keywords: robust control, model predictive and optimization-based control, constrained control, linear systems, feasibility, ensemble systems

1. INTRODUCTION

Large-scale systems composed of a huge number of subsystems, can be found in many application areas, e.g. power systems, quantum systems, mobile robots(Spinelli et al. (2020), Li and Khaneja (2006), Ito et al. (2020)). Typical control strategies for such systems can be classified into decentralized, distributed, hierarchical control and centralized control with a simplified system. Such a simplified system can be obtained by aggregation (Aoki, 1968) or model-reduction methods (Antoulas, 2005).

Ensemble systems, as one special class of such large-scale systems, are composed of structurally identical subsystems driven by the same input. Although there are some related works, constraints on each subsystem are generally not considered, which may be prohibitive in safety-critical applications. Taking such constraints into account is generally challenging, and tremendously limits the available control methods. One possible method is tube-based robust model predictive control with simplified models. It guarantees the constraints of original systems robustly, regardless of the deviation between original systems and simplified models, and demands affordable online computation effort. So far, only a few works, e.g. Dubljevic et al. (2006), Kögel and Findeisen (2015), Lorenzetti et al. (2019), consider this approach in the presence of hard constraints. However, all these mentioned works either assume the full knowledge of the original system's states, which is normally not available in ensemble systems, or is not suitable to guarantee the constraints of each subsystem.

Recently, Aschenbruck et al. (2023) proposed a set-theoretic method based on viability theory (Aubin et al., 2011) to guarantee the safe operation of each subsystem in an ensemble linear system without access to its full state and whole model for online implementation. Inspired by this work, we recast this control problem for the ensemble system as a general moving-horizon control problem involving a reduced order model and the aggregated information of the whole system, and consider adaptive tube formulations for the state constraint satisfactions. For the formulation of the tube, we leverage the results in system's peak-to-peak norm (Bu et al. (1996), Rieber et al. (2006)). Based on the adaptive tubes and the constraints of the ensemble system, we derive adaptive set-based conditions combined with a robust controlled invariant set to ensure the existence of a control law, which guarantees the safe operation of all the subsystems.

The paper is structured as follows: In the next section, the problem setting is stated in the moving-horizon scheme. In Section 3, a method used to approximate the reachable set is presented. Based on this method, the adaptive tube formulation is derived. Following this, a MPC scheme ensuring safety is proposed. In the end, some implementation issues are discussed. In Section 4, a numerical example is shown followed by the conclusion in Section 5.

Notation: Given a matrix F and a non-negative vector g, the associated 0-symmetric polyhedral set is denoted by $\mathcal{P}^0(F,g) := \{x \in \mathbb{R}^n : -g \le Fx \le g\}$. The Euclidean norm of a vector x is denoted by $\|x\|$. Suppose $P = \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix} \in \mathbb{R}^{n\times n}$ with $P_3 \in \mathbb{R}$, we denote $\mathcal{E}(P) := \{x \in \mathbb{R}^{n-1} : \begin{pmatrix} x^T & 1 \end{pmatrix} P \begin{pmatrix} x \\ 1 \end{pmatrix} \le 0\}$ as a subset of $\mathbb{R}^{n-1}$ characterized by P. The translation of the set $\mathcal{X}$ by a vector x is denoted by $\mathcal{X}(x)$. Given a set $\mathcal{S} \subset \mathbb{R}^n \times \mathbb{R}^m$, its projection

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onto the first n coordinates is denoted by $\pi_n(\mathcal{S})$, and its N-ary Cartesian power is denoted by $\mathcal{S}^N$. Given $A_1 \dots A_N$ with appropriate dimensions, we use $\mathrm{col}(A_1, \dots, A_N)$ and $\mathrm{diag}(A_1, \dots, A_N)$ to stack matrices vertically and diagonally respectively. Finally, let I_n denote the n by n identity matrix, and let I and O denote the identity and zero matrices of appropriate sizes respectively.

2. PROBLEM STATEMENT

Consider an ensemble system denoted by $(S_i)_{\mathbb{I}}$, which is composed of N_e discrete time constrained subsystems

$$S_i : \begin{cases} x_i(t+1) = A_i x_i(t) + B_i u(t), x_i(0) = x_{0,i}, \\ u(t) \in \mathcal{U} \subset \mathbb{R}^m, x_i(t) \in \mathcal{X}_i \subset \mathbb{R}^n, t \in \mathbb{N}_0, \end{cases} \tag{1}$$

indexed by $i \in \mathbb{I} := \{1, \dots, N_e\}$, $N_e \in \mathbb{N}$, and driven by the same input $u(t)$. The system matrix A_i of each subsystem S_i is assumed to be Schur stable. The input constraint $\mathcal{U}$ is a bounded polyhedral set and the state constraint $\mathcal{X}_i$ is a polyhedral set. The performance output of $(S_i)_{\mathbb{I}}$, which is used to formulate an online control objective, is the output value of the aggregate function $f(x_1(t), ..., x_{N_e}(t)) := \sum_{i=1}^{N_e} \beta_i x_i(t)$ with $\beta_i \in \mathbb{R}_{\geq 0}$ and $\sum_{i=1}^{N_e} \beta_i = 1$. In this work, we choose w.l.o.g. the function f to be an arithmetic mean, i.e.

$$f(x_1(t), ..., x_{N_e}(t)) = \frac{1}{N_e} \sum_{i=1}^{N_e} x_i(t). \tag{2}$$

The control task is to find a control law associated with an online control objective such that the input and state constraints of all the subsystems given in (1) are satisfied simultaneously. To carry out this control task, a MPC optimization problem using a reduced order model and the performance output is to be formulated, which constitutes *the goal of this work*.

For the MPC formulation, we consider a reduced order model $\bar{S}$, which is named as aggregation model and defined by

$$\bar{S} : \begin{cases} \bar{x}(k+1|t) = \bar{A}\bar{x}(k|t) + \bar{B}u(k|t), \\ \bar{x}(0|t) = f(x_1(t), \dots, x_{N_e}(t)) \in \mathbb{R}^n, \\ u(k|t) \in \mathcal{U} \subset \mathbb{R}^m, k = \{0, \dots, N-1\}, \end{cases} \tag{3}$$

with the predicted sequence $\bar{x}(\cdot|t) := (\bar{x}(1|t), \dots, \bar{x}(N|t))$ and some matrices $\bar{A}$ and $\bar{B}$, which will be specified later. The naming is motivated by the initialization of $\bar{S}$ to the aggregated states of the ensemble system. Besides the input constraints $\mathcal{U}$, the model $\bar{S}$ has to also satisfy some additional input and state constraints, which will be defined in the main result.

For a simpler exposition, we introduce the following definitions in terms of admissibility and reachability.

Definition 1. An input sequence $u(\cdot|t) := (u(0|t), \dots, u(N-1|t)) \in \mathcal{U}^N$ is *admissible* for the ensemble system $(S_i)_{\mathbb{I}}$ if the resulting state sequence satisfies $(x_i(1|t), \dots, x_i(N|t)) \in \mathcal{X}_i^N$ for each i.

Definition 2. Given a discrete dynamic system $x(t+1) = Ax(t) + Bu(t)$ with $y(t) = Cx(t)$, where $u(t) \in \mathcal{U}$ and $x(0)$ is unknown and lies in $\mathcal{X}_0$, the reachable set of $y(\cdot)$ is the set of all vectors $\hat{y}$ for which there exist $x(0) \in \mathcal{X}_0$, $T \in \mathbb{N}_0$, and $u(t) \in \mathcal{U}$ with $t \in [0, T]$ such that $y(T) = \hat{y}$.

3. MAIN RESULTS

3.1 Outer Bounds of Reachable Sets

In this subsection, we propose a generic approach, which is employed for the tube construction, to over-estimate the reachable set of the output of a linear discrete system driven by a pointwise bounded input. In Rieber et al. (2006) and Bu et al. (1996), it was shown that, under the assumption of zero initial condition, the reachable set of the system's output with a unit-peak input can be over-approximated with the upper-bound of the system's peak-induced norm. The inspection of their proofs shows that the restrictive assumption on the initial condition can be readily relaxed to a non-zero bounded initial condition, which is shown in the following theorem.

Theorem 1. Consider a stable discrete-time system

$$x(t+1) = Ax(t) + Bu(t), y = Cx(t)$$

and initial condition $x(0) \in \bigcap_{q=1}^{N_p} \mathcal{E}(Q_q)$, where Q_q's are some symmetric real matrices and $\|u(t)\| \leq 1$ for all $t \in \mathbb{N}_0$. Given $\alpha \in (0, 1)$, let

$$V(\alpha) := \inf_{\lambda_1, \dots, \lambda_{N_p} \geq 0, P, \nu, \gamma} \gamma,$$

s.t.

$$\begin{pmatrix} P & AP & B \\ PA^T & (1-\alpha)P & O \\ B^T & O & \nu I \end{pmatrix} \succ 0, \tag{4}$$

$$\begin{pmatrix} \alpha P & PC^T \\ CP & \gamma I \end{pmatrix} \succ 0, \ \gamma > \nu, \tag{5}$$

$$\begin{pmatrix} \sum_{q=1}^{N_p} \lambda_q Q_q + \left(\begin{smallmatrix} O & O \\ O & \nu \end{smallmatrix}\right) & \left(\begin{smallmatrix} \alpha I \\ O \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} \alpha I & O \end{smallmatrix}\right) & \alpha P \end{pmatrix} \succ 0. \tag{6}$$

Then $\|y(t)\| < V(\alpha)$ for all $t \in \mathbb{N}_0$.

Proof. Suppose the tuple $(\lambda_1, \dots, \lambda_{Np}, P, \nu, \gamma)$ satisfies (4)-(6) with a given α. Applying Schur complement (Boyd et al., 2004) and S-procedure (Yakubovich, 1997) on (6), we obtain

$$x(0) \in \bigcap_{j=1}^{N_p} \mathcal{E}(Q_j) \subset \mathcal{E}(\mathrm{diag}(P^{-1}, -\frac{\nu}{\alpha})), \tag{7}$$

where the non-singularity of P follows from the positive-definiteness of (4). By multiplying (4) from both sides by $\mathrm{diag}(I, P^{-1}, I)$, and then performing Schur complement on the first row and column, we reformulate (4) into

$$\begin{pmatrix} A^T(P)^{-1}A - (1-\alpha)P^{-1} & A^T(P)^{-1}B \\ B^T P^{-1} A & B^T(P)^{-1}B - \nu I \end{pmatrix} \prec 0.$$

Multiplying this inequality from both sides by the vector $\mathrm{col}(x(t), u(t))$, after rearrangement, we obtain

$$V_x(t+1) - (1-\alpha)V_x(t) + \nu\|u(t)\|^2 < 0, \tag{8}$$

where $V_x(t) := x(t)^T(P)^{-1}x(t)$. Since $\|u(t)\| \leq 1$ for all t and $\alpha \in (0, 1)$, (8) implies $V_x(t) - \frac{\nu}{\alpha} < (1-\alpha)^k(V_x(0) - \frac{\nu}{\alpha})$ for $t > 0$, which, together with (7), lead to

$$V_x(t) \leq \frac{\nu}{\alpha}, \forall t \geq 0. \tag{9}$$

Let us multiply the first inequality in (5) from both sides by $\mathrm{diag}(P^{-1}, I)$. The resultant inequality and the second inequality in (5) lead to

$$\mathrm{diag}\left(\begin{pmatrix} \alpha P^{-1} & C^T \\ C & \gamma I \end{pmatrix}, (\gamma - \nu)I\right) \succ 0.$$

Performing Schur complement on this inequality, we get $\mathrm{diag}(\alpha P^{-1} - \frac{1}{\gamma}C^TC, (\gamma-\nu)I) \prec 0$. After multiplying it from both sides by $\mathrm{col}(x(t), u(t))$, we obtain $\alpha V_x(t) + (\gamma - \nu)\|u(t)\|^2 - \frac{1}{\gamma}\|y(t)\|^2 > 0$, which implies $\|y(t)\| < \gamma$ for all t in view of (9) and the assumptions on u and α. In the end, we minimize γ over all the tuples satisfying (4)-(6), which gives $\|y(t)\| < V(\alpha)$.

In the following corollary, which is deduced from Theorem 1, we show that a 0-symmetric polyhedral set can be applied for the point-wise bounding of the output signal. This allows for less conservatism and more efficient computation involving set operation compared to the Euclidean norm used in Theorem 1.

Corollary 2. Given a stable discrete LTI system

$$x(t+1) = Ax(t) + Bu(t), y(t) = Cx(t)$$

with $x(0) \in \bigcap_{q=1}^{N_p} \mathcal{E}(Q_q)$ and $\|Hu(t)\| \leq 1$ for $t \in \mathbb{N}_0$, where H is non-singular. Consider a matrix F having $N_f \in \mathbb{N}$ rows, and denote its s-th row by F_s, with $s \in \{1, \dots, N_f\}$. For each s, let

$$V_s(\alpha_s) := \inf_{\lambda_{s,1},\dots,\lambda_{s,N_p} \geq 0, P_s, \nu_s, \gamma_s} \gamma_s$$

with a given $\alpha_s \in (0,1)$, s.t.

$$\begin{pmatrix} P_s & AP_s & BH^{-1} \\ P_sA^T & (1-\alpha_s)P_s & O \\ H^{-T}B^T & O & \nu_s I \end{pmatrix} \succ 0, \tag{10}$$

$$\begin{pmatrix} \alpha_s P_s & P_s(F_sC)^T \\ F_sCP_s & \gamma_s I \end{pmatrix} \succ 0, \gamma_s > \nu_s, \tag{11}$$

$$\begin{pmatrix} \sum_{q=1}^{N_p} \lambda_{s,q} Q_q + \left(\begin{smallmatrix} O & O \\ O & \nu_s \end{smallmatrix}\right) & \left(\begin{smallmatrix} \alpha_s I \\ O \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} \alpha_s I & O \end{smallmatrix}\right) & \alpha_s P_s \end{pmatrix} \succ 0, \tag{12}$$

where $\left(\begin{smallmatrix} O & O \\ O & \nu_s \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \alpha_s I \\ O \end{smallmatrix}\right)$ are partitioned w.r.t. $\mathrm{col}(x(t), 1)$. Then $y(t) \in \mathcal{P}^0(F, g)$ for all $t \in \mathbb{N}_0$, where $g := (\inf_{\alpha_1} V_1(\alpha_1), \dots, \inf_{\alpha_{N_f}} V_{N_f}(\alpha_{N_f}))^T$.

Proof. Let $\widetilde{u} := Hu$. Substituting $H^{-1}\widetilde{u}$ for u and then using the Theorem 1 repeatedly for all the s, we get $\|F_s y(t)\| < V_s(\alpha_s)$ for all s and t. Since $V_s(\alpha_s)$ is the value of the function V_s at α_s, we minimize the value of V_s over α_s for each s, which gives $\|F_s y(t)\| < \inf_{\alpha_s} V_s(\alpha_s)$ for all t. Since F_s is a vector, $|F_s y(t)| = \|F_s y(t)\|$, we obtain $|F_s y(t)| < \inf_{\alpha_s} V_s(\alpha_s)$ for all s, and hence $y(t) \in \mathcal{P}^0(F, g)$.

The vector g in the set $\mathcal{P}^0(F, g)$ is found by combining the minimization of γ_s under linear matrix inequality constraints for the fixed scalar α_s with a line search over $0 < \alpha_s < 1$ for each s, which can be efficiently solved.

If the input $u(t)$ always lies in some bounded convex polyhedral set, then one can find a smallest ellipsoid $\mathcal{E}(\mathrm{diag}(H^TH, -1))$ covering all the vertices of the polyhedral by solving a convex optimization problem (Boyd et al., 2004), so that the setting for the input in Corollary 2 is fulfilled. As for the initial condition $x(0)$, the multiple quadratic constraints allows us to characterize it more specifically than a single constraint. If $x(0)$ is exactly known, the multiple quadratic constraints can be also used to represent a set containing only one element. It is worth mentioning that, if $x(0)$ is close enough to the origin, then all the ellipsoids characterized by (10) and (11) will contain it, which renders (12) trivial.

3.2 MPC with Simultaneous Constraint Satisfaction

In order to guarantee the constraint satisfaction of each subsystem of the ensemble system, for each system index i we use a state-dependent tube which evolves along the prediction horizon and covers the state deviation between the subsystem S_i and the model $\bar{S}$, i.e. $x_i(k|t) - \bar{x}(k|t)$. Based on these tubes, we derive the state condition for the MPC employing the aggregation model to control the ensemble system with simultaneous constraint satisfaction.

As a first step, we consider the approximation of the reachable set of $N_e^{-1}\big(x_i(\cdot) - x_j(\cdot)\big)$, which can be seen as the output of a linear system described by

$$x(t+1) = \begin{pmatrix} A_i & O \\ O & A_j \end{pmatrix} x(t) + \begin{pmatrix} B_i \\ B_j \end{pmatrix} u(t),$$

$$y(t) = \begin{pmatrix} N_e^{-1}I_n & -N_e^{-1}I_n \end{pmatrix} x(t), u(t) \in \mathcal{U},\ i,j \in \mathbb{I}.$$

By applying Corollary 2 to this linear system with the information about the initial condition of subsystems and a chosen matrix F, we obtain an outer bound of the reachable set, say $\mathcal{P}^0(F, g_{ij})$, which as shown in the following Proposition, is used to determine the the outer bounds of the reachable set of $x_i(k|\cdot) - \bar{x}(k|\cdot)$ at $k = 0$.

Proposition 3. Assume that $\mathcal{P}^0(F, g_{ij})$ contains the reachable set of $N_e^{-1}\ \big(x_i(\cdot) -\ x_j(\cdot)\big)$ for all $i, j \in \mathbb{I}$. Then for each subsystem S_i, $x_i(t) - \bar{x}(0|t) \in \mathcal{P}^0(F, \sum_{j\neq i} g_{ij})$ for all $t \in \mathbb{N}_0$.

Proof. From $x_i(t) = \sum_{j=1}^{N_e} N_e^{-1}x_i(t)$ and the definition of $\bar{x}(0|t)$, it follows that $x_i(t) - \bar{x}(0|t) = \sum_{j\neq i} N_e^{-1}\big(x_i(t) - x_j(t)\big)$, which is contained in $\bigoplus_{j\neq i} \mathcal{P}^0(F, g_{ij})$ under the given assumption. Suppose $a \in \mathcal{P}^0(F, g_{12})$ and $b \in \mathcal{P}^0(F, g_{13})$, then $-g_{12} - g_{13} \leq F(a+b) \leq g_{12} + g_{13}$, i.e. $a + b \in \mathcal{P}^0(F, g_{12} + g_{13})$. By induction, we get $\bigoplus_{j\neq i} \mathcal{P}^0(F, g_{ij}) \subseteq \mathcal{P}^0(F, \sum_{j\neq i} g_{ij})$, which completes the proof.

Note that the bound $\mathcal{P}^0(F, g_{ij})$ turns out to be a zonotope, if the matrix F is non-singular. Hence, due to the properties of zonotopes with Minkowski sums (Kopetzki et al., 2017), the bound is tight in the sense that $\bigoplus_{j\neq i} \mathcal{P}^0(F, g_{ij}) = \mathcal{P}^0(F, \sum_{j\neq i} g_{ij})$.

Theoretically, we can also apply Corollary 2 on the whole ensemble system to obtain an over-approximation of the reachable set of $x_i(\cdot) - \bar{x}(0|\cdot)$. However, this is numerically intractable if the size of the ensemble system is very large. Furthermore, due to the definition of $\bar{x}(0|t)$, Proposition 3 implies also

$$x_i(k|t) - \frac{1}{N_e}\sum_{i=1}^{N_e} x_i(k|t) \in \mathcal{P}^0(F, \sum_{j\neq i} g_{ij}). \tag{13}$$

Following this, the deviation between the subsystem S_i and the aggregation model $\bar{S}$ lies in

$$x_i(k|t) - \bar{x}(k|t) \in \{\frac{1}{N_e}\sum_{i=1}^{N_e} x_i(k|t) - \bar{x}(k|t)\} \oplus \mathcal{P}^0(F, \sum_{j\neq i} g_{ij}).$$

Therefore, we consider the bounding for the discrepancy between the aggregated state of all the subsystems and the state of $\bar{S}$ at $k \geq 1$, which reads

$$\frac{1}{N_e}\sum_{i=1}^{N_e} x_i(k|t) - \bar{x}(k|t) = \frac{1}{N_e}\sum_{i=1}^{N_e} A_i^k x_i(t) - \bar{A}^k \bar{x}(0|t) + \sum_{l=0}^{k-1}(\frac{1}{N_e}\sum_{i=1}^{N_e} A_i^l B_i - \bar{A}^l \bar{B}) u(k-l|t).$$

By exploiting the definition of $\bar{x}(0|t)$, the above expression can be reformulated as

$$\frac{1}{N_e}\sum_{i=1}^{N_e} x_i(k|t) - \bar{x}(k|t) = \bar{w}(k|t) + \Delta_k \bar{x}(0|t), \tag{14}$$

where

$$\bar{w}(k|t) := \sum_{l=0}^{k-1}(\sum_{i=1}^{N_e} N_e^{-1} A_i^l B_i - \bar{A}^l \bar{B}) u(k-l|t) + \sum_{i=1}^{N_e} N_e^{-1}(A_i^k - \bar{A}^k)(x_i(t) - \bar{x}(0|t))$$

and $\Delta_k := \sum_{i=1}^{N_e} N_e^{-1} A_i^k - \bar{A}^k$. Then, using Proposition 3, we obtain

$$\bar{w}(k|t) \in \bar{\mathcal{W}}_k := \bigoplus_{l=0}^{k-1}(\frac{1}{N_e}\sum_{i=1}^{N_e} A_i^l B_i - \bar{A}^l \bar{B})\mathcal{U} \oplus \frac{1}{N_e}\bigoplus_{i=1}^{N_e}(A_i^k - \bar{A}^k)\mathcal{P}^0(F, \sum_{j\neq i} g_{ij}), \tag{15}$$

as long as the input is always restricted in $\mathcal{U}$. Finally, the tube covering the deviation between the subsystem S_i and the aggregation model is given by

$$x_i(k|t) - \bar{x}(k|t) \in \mathcal{P}^0(F, \sum_{j\neq i} g_{ij}) \oplus \bar{\mathcal{W}}_k(\Delta_k \bar{x}(0|t)) \tag{16}$$

for $k \geq 1$. By exploiting (16) together with Proposition 3, we specify the state conditions on the aggregation model, under which the constraints on the current and predicted state of the ensemble system are simultaneously satisfied.

Proposition 4. Assume that the state of the aggregation model $\bar{S}$ at time t is constrained by

$$\bar{x}(k|t) \in \mathcal{V} \ominus \bar{\mathcal{W}}_k(\Delta_k \bar{x}(0|t)) \tag{17}$$

with $\mathcal{V} := \bigcap_{i=1}^{N_e}\big(\mathcal{X}_i \ominus \mathcal{P}^0(F, \sum_{j\neq i} g_{ij})\big)$ for $k \in \{1,\dots,N\}$ and $\bar{x}(k|t) \in \mathcal{V}$ for $k = 0$. Then $x_i(t) \in \mathcal{X}_i$ for all the subsystems and every input sequence $u(\cdot|t) \in \mathcal{U}^N$ enforcing (17) is admissible for the ensemble system.

Proof. Following (16) and the anti-extensive property of opening (Heijmans, 1987), at each step $k \in \{1,\dots,N\}$, $x_i(k|t) \in \mathcal{X}_i$ for i-th subsystem if

$$\bar{x}(k|t) \in \mathcal{X}_i \ominus \big(\mathcal{P}^0(F, \sum_{j\neq i} g_{ij}) \oplus \bar{\mathcal{W}}_k(\Delta_k \bar{x}(0|t))\big).$$

Hence $x_i(k|t) \in \mathcal{X}_i$ for all $i \in \mathbb{I}$, if $\bar{x}(k|t)$ is restricted to

$$\bigcap_{i=1}^{N_e}\Big(\mathcal{X}_i \ominus \big(\mathcal{P}^0(F, \sum_{j\neq i} g_{ij}) \oplus \bar{\mathcal{W}}_k(\Delta_k \bar{x}(0|t))\big)\Big),$$

which is exactly (17) in virtue of algebraic properties of Minkowski operations (Heijmans, 1987), The result for $k = 0$ follows directly from Proposition 3 and the anti-extensivity.

In Proposition 4, with a given $\bar{x}(0|t)$, it is implicitly assumed that there exists an admissible input sequence, which is generally not valid without further treatments. Furthermore, recursive feasibility can not be concluded. In the following theorem, by imposing tighter constraints on $\bar{S}$, we show how to generate an admissible input sequence recursively for the ensemble system under an easily verifiable assumption. To this end, let us first define a feasible set $\mathcal{S}$ in terms of the tupel $(\bar{x}(0|t), \bar{x}(1|t))$ by

$$\begin{aligned}\mathcal{S} = \big\{&(\bar{x}(0), \bar{x}(1)) \in \mathcal{V} \times \mathbb{R}^n : \exists u(\cdot) \in \mathcal{U}^{N-1},\ s.t.\\ &\bar{x}(k+1) + \Delta_{k+1}\bar{x}(0) \in \mathcal{V} \ominus \bar{\mathcal{W}}_{k+1},\\ &\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k), \forall k \in \{1,\dots,N-1\}\big\}\end{aligned}$$

with $\mathcal{V} \subset \mathbb{R}^n$ defined in (17). This feasible set concerns only the constraints at $k \geq 2$. From this set $\mathcal{S}$, we extract a mixed constraint set on the state $\bar{x}(0|t)$ and the input $u(0|t)$ as follows

$$\mathcal{S}_{\bar{x},u} = \{(\bar{x}, u) \in \mathbb{R}^n \times \mathbb{R}^m : (\bar{x}, \bar{A}\bar{x} + \bar{B}u) \in \mathcal{S}, u \in \mathcal{U}\}.$$

Finally, based on $\mathcal{S}_{\bar{x},u}$, we derive a set $\bar{\mathcal{C}}$ for $\bar{x}(0|t)$

$$\begin{aligned}\bar{\mathcal{C}} = \{&\bar{x}_0 \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ with } (\bar{x}_0, u) \in \mathcal{S}_{\bar{x},u},\\ &s.t.\ \forall d \in \bar{\mathcal{W}}_1,\ (\bar{A} + \Delta_1)\bar{x}_0 + \bar{B}u + d \in \bar{\mathcal{C}}\},\end{aligned}$$

which is a robust controlled invariant set within $\pi_n(\mathcal{S}_{\bar{x},u})$, see Anevlavis et al. (2021) for details.

Theorem 5. Assume $\bar{x}(0|0)$ is located in $\bar{\mathcal{C}}$. An admissible input sequence $u^*(\cdot|t) \in \mathcal{U}^N$ for the ensemble system $(S_i)_{\mathbb{I}}$ can be generated for time $t = 0$ and recursively for all future times by the following MPC scheme

$$\begin{aligned}\min_{u(\cdot|t)}\ & J(\bar{x}(0|t), u(\cdot|t))\\ s.t.\ & (3), (\bar{x}(0|t), u(0|t)) \in \mathcal{S}_{\bar{x},u},\\ & \bar{x}(1|t) \in \bar{\mathcal{C}} \ominus \bar{\mathcal{W}}_1(\Delta_1 \bar{x}(0|t)),\\ & \bar{x}(k|t) \in \mathcal{V} \ominus \bar{\mathcal{W}}_k(\Delta_k \bar{x}(0|t)), k \in \{2,\dots,N\},\end{aligned} \tag{18}$$

where the open-loop cost J is subject to the online control objective. Furthermore, the resulting feedback control law $\mu(\bar{x}(0|t)) = u^*(0|t)$ ensures the constraint satisfaction of the controlled ensemble system for all times.

Proof. $\bar{x}(0|0) \in \bar{\mathcal{C}}$ implies that there exists $u(0|0) \in \mathcal{U}$, such that $\bar{x}(1|0) \in \bar{\mathcal{C}} \ominus \bar{\mathcal{W}}_1(\Delta_1(0|0))$ and $(\bar{x}(0|0), u(0|0)) \in \mathcal{S}_{\bar{x},u}$ in virtue of (3) and the definitions of $\bar{\mathcal{C}}$. Furthermore, since $\bar{\mathcal{C}} \subseteq \pi_n(\mathcal{S}_{\bar{x},u}) \subseteq \pi_n(\mathcal{S}) \subseteq \mathcal{V}$, we obtain $\bar{x}(1|0) \in \mathcal{V} \ominus \bar{\mathcal{W}}_1(\Delta_1(0|0))$. From the definition of $\mathcal{S}_{\bar{x},u}$ and $\mathcal{S}$, the aforementioned $(\bar{x}(0|0), u(0|0)) \in \mathcal{S}_{\bar{x},u}$ implies that there exists an input sequence $(u(1|0),\dots,u(N-1|0)) \in \mathcal{U}^{N-1}$ such that $\bar{x}(k|0) \in \mathcal{V} \ominus \bar{\mathcal{W}}_k(\Delta_k \bar{x}(0|0))$ for all $k \in \{2,\dots,N\}$. Hence there exists $u^*(\cdot|0) \in \mathcal{U}^N$ which minimizes the control objective J and renders (17) valid at time $t = 0$. This yields that $u^*(\cdot|0)$ is admissible for the ensemble system at $t = 0$ in view of Proposition 4. From the definition of $\bar{x}(0|t)$, (14) and $x_i(1|t) = x_i(0|t+1)$, it follows that $\bar{x}(0|1) - \bar{x}(1|0) = \sum_{i=1}^{N_e} N_e^{-1} x_i(1|0) - \bar{x}(1|0) = \bar{w}(1|0) + \Delta_1 \bar{x}(0|0) \in \mathcal{W}_1(\Delta_1 \bar{x}(0|0))$. Since there exists $u^*(0|0)$ enforcing $\bar{x}(1|0) \in \bar{\mathcal{C}} \ominus \mathcal{W}_1(\Delta_1 \bar{x}(0|0))$, we obtain $\bar{x}(0|1) \in \bar{\mathcal{C}}$ in view of the anti-extensivity. Hence, there exists an admissible input sequence for the ensemble system at $t = 1$. Repeating the above steps recursively, we show that an admissible input sequence $u^*(\cdot|t)$ exists recursively for $t = 0$ and all future times. Since $\bar{x}(0|0) \in \mathcal{V}$, and there exists a feedback control law $u^*(0|t) \in \mathcal{U}$ ensuring $\bar{x}(0|t+1) \in \mathcal{V}$, the state constraints of the controlled ensemble system are always satisfied in virtue of Proposition 4.

It is worth mentioning that if the difference among the dynamics of subsystems is large, the set $\bar{\mathcal{C}}$ can be quite small or even empty, which renders the proposed method non-applicable.

3.3 Discussion

In the following, we discuss some issues appearing during applying the proposed approach.

Computation of $\mathcal{S}$ for large N The computation of the feasible set $\mathcal{S}$ in a patch way entails the projection from the subspace of $\mathcal{V} \times \mathbb{R}^n \times \mathcal{U}^{N-1}$ onto the first $2n$ coordinates, which is numerically difficult if the prediction horizon is large. A remedy is to compute the set $\mathcal{S}$ in a recursive way, which is illustrated in the following.

Algorithm: Iterative computation of $\mathcal{S}$

Initialize $\mathcal{S}_{N+1} = \{(\bar{x}_0, \hat{x},) \in \mathcal{V} \times \mathbb{R}^n\}$.
For $k = N$ to 2:
$\mathcal{S}_k = \{(\bar{x}_0, \hat{x}) : \exists u \in \mathcal{U}, s.t.$
$\bar{A}\hat{x} + \bar{B}u + \Delta_k \bar{x}_0 \in \mathcal{V} \ominus \bar{\mathcal{W}}_k, (\bar{x}_0, \bar{A}\hat{x} + \bar{B}u) \in \mathcal{S}_{k+1}\}$.
Return
$\mathcal{S} = \mathcal{S}_2$.

Consequently, the subspace, in which the projection is performed in each step, is just the subspace of $\mathcal{V} \times \mathbb{R}^n \times \mathcal{U}$.

Choice of the aggregation model $\bar{S}$ As defined in the problem setting, the aggregation model is initialized to the aggregated states of all the subsystems at the beginning of each horizon. Hence, if the prediction horizon is not large, the choice of $\bar{S}$ will not exhibit a major influence on the restrictiveness of the constraints in Theorem 5, as long as the dynamic of $\bar{S}$ differs not tremendously from that of the subsystems, which implicitly requires a certain degree of similarity of all subsystems. In practice, a sensible choice of the dynamic of $\bar{S}$ can be

$$\bar{A} = \frac{1}{N_e}\sum_{i=1}^{N_e} A_i, \bar{B} = \frac{1}{N_e}\sum_{i=1}^{N_e} B_i, \tag{19}$$

which turns out to be the model derived by aggregation in Aoki (1968). If the prediction horizon is large, then the deviation between the ensemble system and $\bar{S}$ chosen as (19) may be large, especially when $\bar{S}$ is unstable. In such a case, we can exploit Theorem 1 to find a stable linear model, whose trajectory is similar to that of the aggregated state of the ensemble system. Since the order of $\bar{S}$ is restricted to n, a nonconvex constraint arises. By imposing some convex structure constraints on some decision variable matrices, we can get rid of the aforementioned nonconvex constraint, however, with a compromise of larger suboptimality. The obtained system has then maximal n observable states, which ensures the requirement on the system order. In the end, the system corresponding to the minimal $V(\alpha)$ over all the α can be chosen as the aggregation model. A similar idea can be found in Geromel et al. (2005).

4. NUMERICAL EXAMPLE

In this section, we demonstrate the proposed MPC scheme in a numerical example, which is taken from Aschenbruck et al. (2023). The ensemble system $(S_i)_{\mathbb{I}}$ is modeled by:

$$x_1(t+1) = \begin{pmatrix} 0.92 & 0.09 \\ -0.09 & 0.87 \end{pmatrix} x_1(t) + \begin{pmatrix} 0.0195 \\ 0.44 \end{pmatrix} u(t),$$
$$x_2(t+1) = \begin{pmatrix} 0.93 & 0.09 \\ -0.09 & 0.86 \end{pmatrix} x_2(t) + \begin{pmatrix} 0.02 \\ 0.46 \end{pmatrix} u(t),$$
$$x_3(t+1) = \begin{pmatrix} 0.94 & 0.09 \\ -0.09 & 0.85 \end{pmatrix} x_3(t) + \begin{pmatrix} 0.02 \\ 0.44 \end{pmatrix} u(t),$$

with the state constraints given by $\mathcal{X}_i = \{x_i \in \mathbb{R}^2 : -2 \le x_{i,1} \le 2\}$ for $i \in \mathbb{I} = \{1, 2, 3\}$ and the input constraint $\mathcal{U} = \{u \in \mathbb{R} : -1 \le u \le 1\}$. In oder to let the subsystems approach the target $\bar{x}_T$, we choose the open-loop cost in MPC to be

$$J(\bar{x}(0|t), u(\cdot|t)) = \sum_{k=1}^{N} \|\bar{x}(k|t) - \bar{x}_T\|^2$$

with a prediction horizon $N = 10$. For each prediction horizon, $\bar{x}$ is updated according to the aggregation model chosen as (19). The matrix F in Proposition 3 is chosen to be the identity. The employed MPC scheme with $\bar{S}$ chosen as (19) reads

$$\min_{u(\cdot|t)} J(\bar{x}(0|t), u(\cdot|t))$$
$$s.t.\ (3), (2), (\bar{x}(0|t), u(0|t)) \in \mathcal{S}_{\bar{x},u}, \bar{x}(1|t) \in \bar{\mathcal{C}} \ominus \bar{\mathcal{W}}_1,$$
$$\bar{x}(k|t) \in \mathcal{V} \ominus \bar{\mathcal{W}}_k(\Delta_k \bar{x}(0|t)), k \in \{2, \ldots, N\}.$$

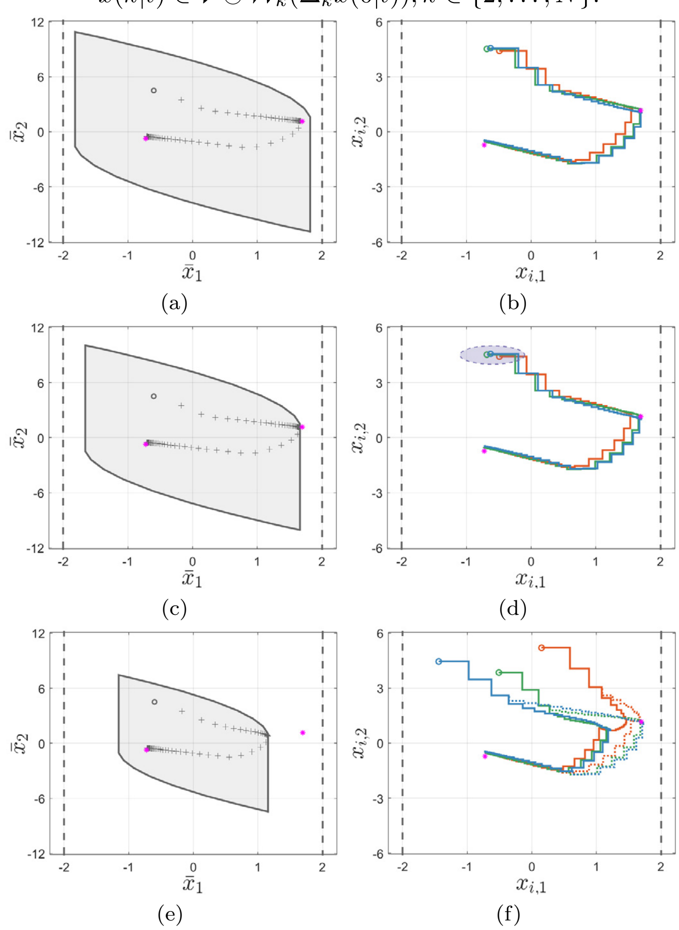

Fig. 1. Simulation results of $\bar{S}$ in (a), (c) and (e) as well as $(S_i)_{\mathbb{I}}$ in (b), (d) and (f). Scenario (I): (a) and (b); Scenario (II): (c) and (d); Scenario (III): (e) and (f). $\bar{x}(1|t)$ (black plus), $\bar{\mathcal{C}} \ominus \bar{\mathcal{W}}_1$ (gray).

In the simulation, the target state is changed from $(1.69, 1.15)^T$ to $(-0.72, 0.49)^T$ after 33 second. Furthermore, the following 3 scenarios are considered: (I) all $x_i(0)$'s are close to $\bar{x}(0|0) = (-0.6, 4.5)^T$ and their exact position is available for Corollary 2. (II) all $x_i(0)$'s are close as in (I), but only known to lie in $\mathcal{X}_{\bar{x}(0|0)} := \{x : \|x - \bar{x}(0|0)\| \leq 0.25\}$ for Corollary 2. (III) all $x_i(0)$'s are more separated with the same $\bar{x}(0|0)$, and their exact position is available for Corollary 2. The respective simulation results are shown in Fig. 1, where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are depicted in red, blue, and green respectively and marked additionally with a circle at $t = 0$. For comparison purposes, a nominal MPC with full model and state is designed. The resulted system trajectories are shown with dashed lines in scenario (III). For other scenarios, the results are not shown due to minor differences. The target states and $\bar{x}(0|0)$ are marked with magenta asterisk and black circle respectively. The set $\mathcal{X}_{\bar{x}(0|0)}$ in (II) is depicted in purple. The algorithm from Anevlavis et al. (2021) is used to compute $\bar{\mathcal{C}}$. For the other set computation and MPC design, the toolbox MPT3 (Herceg et al., 2013) is used. Fig. 1 shows, that all subsystems stay in their own safety

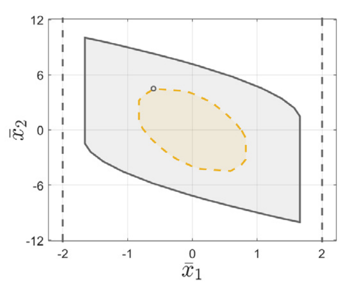

Fig. 2. The set for $\bar{x}(0|0)$ (yellow), in which the set of constraint for $\bar{x}(1|t)$ derived with the $\mathcal{X}_{\bar{x}(0|0)}$ contains the set (gray) derived with $\bar{x}(0|0) = (-0.6, 4.5)^T$ (black circle)

area when approaching the target. Furthermore, the closer the initial conditions $x_i(0)$'s, and the more specific the position of $x_i(0)$'s, the larger the set of constraint for $\bar{x}(1|t)$. This makes sense, since the output bound determined by Corollary 2 tends to be larger with a larger set characterizing the initial condition. Fig. 2 indicates that a larger set of constraint for $\bar{x}(1|t)$ in case of uncertain initial conditions can be achieved with other aggregated initial conditions, which illustrates the robustness of the approach against the change of initial conditions.

5. CONCLUSION AND OUTLOOK

In this work, we propose a MPC scheme using an aggregation model to enforce the safety operation of each subsystem in a large-scale ensemble system. To this end, we adopt an adaptive tube formulation to cover the deviation between the simplified model and the ensemble system. Furthermore, by exploiting the controlled robust invariant set, the formulated MPC scheme is recursively feasible and initially feasible. In practice, process and measurement errors are often not negligible, and the subsystem may be nonlinear and not exactly known. These issues will be considered as the focus of future research.

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